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Completeness in the Set of Real numbers.

The property of Completness (or Order-Completness) which is possessed by R and not by Q. Together with Ordered field property it characterises R i.e. the set of real numbers is the only set which is a Complete Ordered Field.

Order-Completeness in R.

Every non-empty set of real numbers which is bounded above has the supremum (or the least upper bound) in R.

In other words. The set of upper bounds of a non-empty set of real numers bounded above has the smallest ~~number~~ member.

If S is a set of real numbers which is bounded above, then by considering the set $T = \{x \mid -x \in S\}$

We can state completeness property in the alternate form as:
Every non-empty set of real numbers which is bounded ~~above~~ below has the infimum (or g. l. b.) in R

Archimedean Property of Real numbers.

Theorem 1. The real number field is Archimedean, i.e. if a and b be any two positive real numbers then there exists a positive integers n such that $na > b$

Proof Let a, b be any two positive real numbers. Let us suppose if possible that for all positive integers $n (\in I^+)$, $na \leq b$

Thus the set $S = \{na : n \in I^+\}$ is bounded above, b being an upper bound. By the completeness property of the ordered field of real numbers, set S must have the supremum M

$$na \leq M \quad \forall n \in I^+$$

$\Rightarrow (n+1)a \leq M, \forall n \in I^+$

$\Rightarrow na \leq M - a, \forall n \in I^+$

i.e. $M - a$ is an upper bound of S.

Thus a number, $M - a$ less than the supremum M (l.u.b) is an upper bound of S; which is a contradiction and hence our supposition is wrong.

Hence the theorem.